Lecture 4:

Existence of basis

For a finite-dimensional vector space, the basis can be constructed as follows:

$$\vec{v}_1 \neq \vec{0}$$

$$\{\vec{v}_1\} \leftarrow \text{Linear independent}$$

$$\downarrow$$

$$\{\vec{v}_1, \vec{v}_2\} \leftarrow \text{Attach one more vector } \vec{v}_2 \ni \{\vec{v}_1, \vec{v}_2\} \text{ is linearly independent}$$

$$\downarrow$$

$$\vdots$$

$$\{\vec{v}_1, \dots, \vec{v}_n\} \leftarrow \text{must stop at some point.}$$

Constructive proof for the existence of basis.

<u>Example:</u> Consider $F^\infty = \{(a_1, a_2, \dots) : a_j \in F\}$.

Let $S_i = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_i\}$

$\vec{e}_1 = (1, 0, \dots 0)$, $\vec{e}_2 = (0, 1, 0, \dots)$

Then: $S_1 \subset S_2 \subset \dots \subset S_i \subset \dots$

Let $S = \bigcup_i S_i$, which is linearly independent.

Obviously $\mathrm{span}(S) \neq F^\infty$.

So, we can find $\vec{v} \notin \mathrm{span}(S) \ni S \cup \{\vec{v}\}$ is linearly independent.

We can repeat the process.

Question: will the process stop??

Zorn's Lemma

Let S be a partially ordered set. If every chain of S has an upper bound in S, then S contains a maximal elements.

5 definitions

1. Partially ordered
2. Totally ordered
3. Chain
4. Maximal element
5. Upper bound

Definition 1: (Partially ordered) A partially ordered on a (non-empty) set S is a binary relation on S, denoted $\leq$, which satisfies:

- for $\forall s \in S$, $s \leq s$
- if $s \leq s'$ and $s' \leq s$, then $s = s'$
- if $s \leq s'$ and $s' \leq s''$, then $s \leq s''$.

Example:

1. With the usual $\leq$ on $\mathbb{R}$. $\mathbb{R}$ is partially ordered.

2. On $\mathbb{N}$, define $a \leq b$ if $a \mid b$. (a divides b)

   Then, $\mathbb{N}$ is partially ordered.

3. Let $\mathcal{C}$ be the collection of subsets of a set S.

   Define $\leq$ by $\underset{\substack{\cap \\ S}}{A} \leq \underset{\substack{\cap \\ S}}{B}$ if $A \subseteq B$.

   Then, $\mathcal{C}$ is a partially ordered set.

Remark: We do not assume all pairs of elements in a partially ordered set are comparable under $\leq$.

Definition 2: If every elements in a partially ordered set S is comparable under $\leq$, then S is called a totally ordered set.

Example: • $\mathbb{R}$ under usual $\leq$ is totally ordered

Definition 3: A chain is a collection of elements in S satisfying: if $s_1 \in S$ and $s_2 \in S$, then either $s_1 \leq s_2$ or $s_2 \leq s_1$.

• Let $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \subseteq \cdots$ be a chain of subsets of S.

Then: $\mathcal{C} = \{C_1, C_2, \dots, C_n, \dots\}$ is a totally ordered set under $\subseteq$.

For simplicity, we may consider countable chain $s_1 \leq s_2 \leq \cdots$ in our discussion for the ease of explanation (not necessarily countable)

Definition 4: A maximal element m of a partially ordered set S is defined as follows: for $\forall s \in S$ to which m is comparable, $s \leq m$.

Remark: • This does not mean $s \leq m$ for $\forall s \in S$.

• A partially ordered set can have many maximal elements.

Definition 5 An upper bound $\tilde{C}$ of a chain $\{C_i\}_{i \in I}$ is that for all $i \in I$, $C_i \leq \tilde{C}$.

Theorem: Every vector space has a basis.

Proof: Let $\mathcal{C}$ be the collection of all linearly independent subsets of V.

For any chain $\{S_i\}_{i \in I}$ (We may consider a countable chain $S_1 \subset S_2 \subset \dots$ for easier interpretation.)

$\bigcup_{i \in I} S_i$ is also linearly independent. $\therefore \bigcup_i S_i \in \mathcal{C}$.

By Zorn's lemma, $\exists$ maximal linearly independent set M.

We claim that $\text{span}(M) = V$.

If not, $\exists\, \vec{v} \in V \ni \vec{v} \notin \text{Span}(M)$.

Then: $M \cup \{\vec{v}\}$ is linearly independent.

But $M \subset M \cup \{\vec{v}\}$. Contradiction to Zorn's lemma.

$\therefore$ ① $\text{Span}(M) = V$

② M is L.I.

$\Rightarrow M$ is a basis.